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# Numerical Exploration of a Network of Nonlocally Coupled Josephson Junction Spurred by Wien Bridge Oscillators

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**ABSTRACT** This paper is devoted to the numerical investigation of the collective dynamics of nonlocally coupled Josephson junctions (JJ) spurred by Wien bridge oscillators (JJSWBOs). A single JJSWBO displays monostable and bistable chaotic characteristics, bistable period-3-oscillations and coexistence between regular and chaotic characteristics. In order to investigate the collective dynamics of the considered network, the local dynamics is set in the bistable regime which includes periodic and chaotic dynamics. The initial conditions for the network dynamics are specially prepared such that the networks units are randomly distributed on these two types of dynamical behaviors. As the value of the coupling strength increases, the dynamics of the network of JJSWBOs undergoes a transition from complete incoherence to coherent travelling waves via a chimera state.

**KEYWORDS**
Josephson junction
Wien bridge oscillator
Chaotic and coexisting characteristics
Nonlocal coupling
Coherent and incoherent behaviors
Chimera state

## INTRODUCTION

Many phenomena in nature result from the intricate interaction between the dynamical characteristics of tremendous elements. For example, the cognitive function of the brain and certain brain disorders such as epilepsy and Parkinson's disease are the macroscopic manifestation of the interaction between the neurons. The most investigated collective dynamical behavior of interacting elements is synchronization, which can be simply defined as the entrainment of rhythms of these interacting elements (Pikovsky *et al.* 2001). The study of synchronization is of paramount importance since it is ubiquitous in the functioning of natural and manufactured systems.

Among the plethora of synchronization phenomena, the chimera state has attracted a lot of attention since its discovery in the early 2000s (Kuramoto and Battogtokh 2002). The chimera state is an intriguing partial synchronization state arising in networks of symmetrically coupled identical systems and characterized by coexisting domains of coherent and incoherent systems. The chimera state was first discovered and mathematically studied in networks of nonlocally coupled phase oscillators (Kuramoto and Battogtokh 2002; Abrams and Strogatz 2004). Since then, a plethora of related patterns have been unveiled in diverse networks of diverse systems, including limit-cycle oscillators, chaotic oscillators, excitable systems, and multistable systems (for a review, see (Zakharova 2020; Parastesh *et al.* 2021; Schöll 2016)). In addition, chimera states were uncovered in experimental setups (Hagerstrom *et al.* 2012; Tinsley *et al.* 2012).

The study of collective dynamics of coupled systems of Josephson junctions is a subject that has been tackled in recent years. Many works on synchronization in arrays of Josephson junctions were carried out (Ngongiah *et al.* 2023; Ramakrishnan *et al.* 2021; Filatrella *et al.* 1992). Very recently, a few works reported the observation of chimeralike states in networks of Josephson junctions (Mishra *et al.* 2017a,b). Thus, the investigation of chimera states in coupled systems of Josephson junctions remains an open problem.

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Besides, bi- or multi-stability has proved to be a key ingredient for the formation of chimera states (Yeldesbay *et al.* 2014). Diverse chimera and chimeralike states have been observed in diverse networks of multistable systems (Dudkowski *et al.* 2016; Mishra *et al.* 2015; Shepelev *et al.* 2017; Lei *et al.* 2022; Ramamoorthy *et al.* 2022; Zhang *et al.* 2024; Nganso *et al.* 2023). In the present study, motivated by the fact that for some parameter values, the JJSWBO is multistable, the authors of this paper intend to investigate chimera states in a network of many JJSWBOs.

## DYNAMICAL CHARACTERISTICS OF JJSWBO AND NETWORK OF NONLOCALLY COUPLED JJSWBOS

The JJSWBO designed by coupling through, a gain a resistive-capacitive shunted JJ is described by the given set of equations (Sriram *et al.* 2023):

$$\frac{dV_1}{dt} = -\frac{k}{2}\left[(|V_1+k| - |V_1-k|) - V_2 - \beta_R V_1 - \alpha\sin(\phi)\right], \quad (1a)$$

$$\frac{dV_2}{dt} = \beta_c\left[\frac{k}{2}(|V_1+k| - |V_1-k|) - V_2\right], \quad (1b)$$

$$\frac{d\phi}{dt} = \gamma V_1. \quad (1c)$$

where $v_1$ and $v_2$ are the voltages, $\phi$ the phase difference of the JJ, and $k$, $\alpha$, $\beta_R$, $\beta_c$, and $\gamma$ are the system parameters. The dynamical characteristics of JJSWBO are illustrated by the phase plane projections of Fig. 1.

Monostable, stable, and chaotic characteristics of different presentations are shown in Fig. 1(a-e), bistable period-3 oscillations for specific control parameters and varying recipient states as depicted in Fig. 1(f), bistable chaotic attractors for chosen parameters and different initial states as elaborated in Fig. 1(g), coexistence between regular and chaotic characteristics for some fixed values of control parameters and different starting conditions as captured in Fig. 1(h1 and h2). The coexistence of periodic and chaotic characteristics shown in Fig. 1(h1 and h2) is further illustrated by the basin of attraction in Fig. 2. The attraction basin of Fig. 2 shows the regions of chaotic characteristics painted in magenta and the regions of periodic characteristics painted in green.

Searching for the chimera states, a lot of coupling functions is tested and it is found that chimera states emerge in networks of nonlocally coupled JJSWBOs with direct and cross terms in the coupling function which is applied on the $v_1$-variable differential equation. A large network of nonlocally coupled JJSWBOs is considered, where each network unit is coupled to its $2P$ nearest neighbors, with $P$ neighbors at either direction around the considered unit. Thus, the dynamics of a given node $j$ of the considered network is described by the following set of differential equations

$$\begin{aligned}\frac{dv_j^1}{dt} =& -\frac{k}{2}\left(\left|v_j^1+k\right| - \left|v_j^1-k\right|\right) - v_j^2 - \beta_R v_j^1 \\ & - \alpha\sin(\phi^j) + \sigma\sum_{i=j-P}^{j+P}\left[a(v_i^1 - v_j^1) + b(\phi^i - \phi^j)\right] \end{aligned} \quad (2a)$$

$$\frac{dv_j^2}{dt} = \beta_c\left[\frac{k}{2}(\left|v_j^1+k\right| - \left|v_j^1-k\right|) - v_j^2\right], \quad (2b)$$

$$\frac{d\phi_j}{dt} = \gamma v_j^1, \quad (2c)$$

where $j = 1, 2, \ldots, N$, $N$ is the size of the network, $\sigma_1 = a\sigma$ and $\sigma_2 = b\sigma$ are real numbers that denote the coupling strengths. In addition, periodic boundary conditions are assumed. For the formation of chimera states in the network, $\alpha$ must be positive and $\beta$ negative. The combination of direct and cross couplings proved to induce the occurrence of chimeralike states and other symmetry-breaking states, namely, oscillation death states in a network of globally coupled JJs (Mishra *et al.* 2017b). It was shown in (Nganso *et al.* 2023) that such a competitive coupling is also responsible for the formation of chimera and solitary states in networks of van der Pol oscillators.

In order to obtain chimera states, initial conditions very often play a pivotal role Zakharova (2020); Parastesh *et al.* (2021); Schöll (2016). The historically first observed chimera state was obtained with specially prepared initial conditions (Kuramoto and Battogtokh 2002). In the present paper, if the dynamics of all network nodes are launched from the basin of attraction of the periodic attractor (green strips in Fig. 2), they converge to the same dynamical state, and even a phase lag between the periodic oscillations of the different nodes is not observed. Thus, in the perspective of investigating symmetry-breaking states, we managed to get initial conditions disseminated randomly in the basins of attraction of the periodic and chaotic attractors. To do so, we considered $\phi_j(0) = 8$ for all nodes (in accordance with the result displayed in Fig. 2), $v_j^1(0)$ and $v_j^2(0)$ are randomly distributed in $[-0.25, 0.25]$ and $[-6, -5]$, respectively.

The set of equations describing the considered network [i.e., Eq. (2)] is solved numerically with the above initial conditions, the parameters used in Fig. 2 for the local dynamics, and $a = 0.08$, $b = -0.8$, and $P = 20$ for the coupling term. The coupling parameter $\sigma$ is considered as a control parameter for the investigation of the collective dynamics of the coupled oscillators. The results of the numerical simulation of Eq. (2) are shown in Fig. 3 for an increasing value of the coupling parameter $\sigma$. Apart from spatiotemporal plots of the different collective dynamical states obtained when varying $\sigma$, the corresponding mean phase velocity profiles $\{\omega_j\}, j = 1, 2, \ldots, N$ are also displayed. The mean phase velocity $\omega_j$ of a given network node $j$ is evaluated numerically from the time series of $v_j^1$ as follows:

$$\omega_j = \frac{2\pi M_j}{\Delta t}, \quad (3)$$

where $M_j$ is the number of oscillations performed by $v_j^1$ during the time interval $\Delta t$ Omelchenko *et al.* (2013).

The mean phase velocity is a quantitative measure widely used for the characterization of certain chimera states and solitary states. Indeed, in case of phase chimera, the mean phase velocity profile presents flat pieces which correspond to coherent parts of the chimera pattern and arc-shaped pieces which correspond to incoherent parts (Kuramoto and Battogtokh 2002; Abrams and Strogatz 2004). For a solitary state, the mean phase velocity profile is overall flat and it involves some singularities which break away from the flat background—the flat background corresponds to the common frequency of synchronous nodes and the singularities correspond to solitary nodes (Rybalova *et al.* 2019).

Fig. 3(a) confirms that the isolated nodes ($\sigma = 0$, i.e., the network nodes are uncoupled) are randomly distributed on two types of dynamical behaviors: the periodic and chaotic dynamics. All the oscillators exhibiting the periodic behavior have the same frequency and the oscillators exhibiting chaotic dynamics have disparate frequencies [see $\{\omega_j, j = 1, 2, \ldots, N\}$ in Fig. 3(a)]. As the value of the coupling parameter $\sigma$ increases, the dynamics of the network switches from complete incoherence [Fig. 3(b)] to coherent

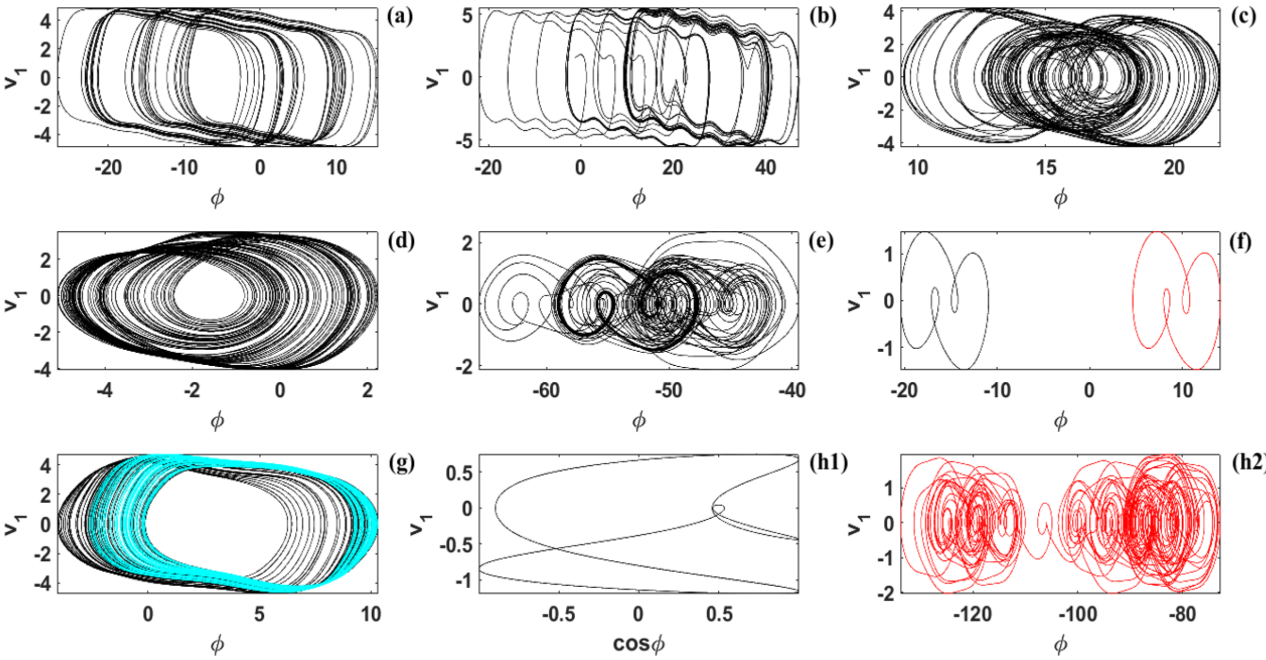


**Figure 1** Evolution of JJSWBO described by system (1) in the plane $(v_1, \phi)$ for varying $\alpha$, $\beta_R$, and $\gamma$: **(a)** $\alpha = 0.2$, $\beta_R = 0.1$, $\gamma = 0.9$, **(b)** $\alpha = 0.9$, $\beta_R = 0.1$, $\gamma = 0.9$, **(c)** $\alpha = 0.45$, $\beta_R = 0.468$, $\gamma = 0.9$, **(d)** $\alpha = 0.45$, $\beta_R = 0.535$, $\gamma = 0.9$, **(e)** $\alpha = 0.45$, $\beta_R = 0.8$, $\gamma = 4.05$, **(f)** $\alpha = 0.45$, $\beta_R = 0.258$, $\gamma = 0.9$, and **(h1, h2)** $\alpha = 0.45$, $\beta_R = 0.8$, $\gamma = 5.037$. The remaining system parameters are limited to $k = 3.2$ and $\beta_c = 2.5$. The recipient conditions include: $(v_1(0), v_2(0), \phi(0)) = (0, 0, 1)$ for black lines, $(v_1(0), v_2(0), \phi(0)) = (0, 8, 0)$ for red lines, and $(v_1(0), v_2(0), \phi(0)) = (1, 1, 1)$ for cyan lines.

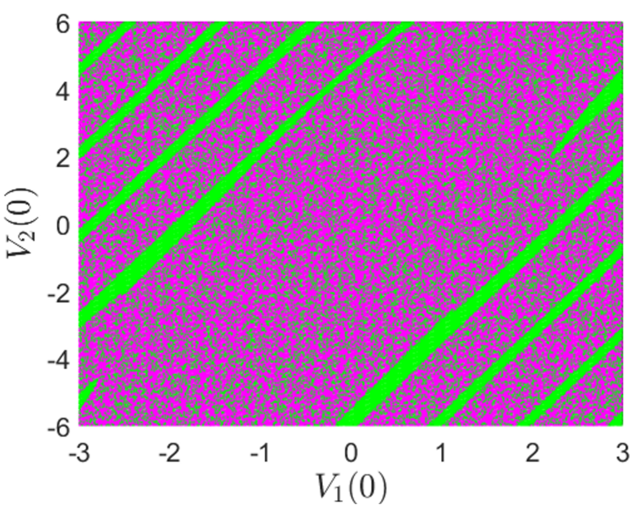


**Figure 2** Overview of the basin of attraction of system (1) in the coordinate space for $(-3 \leq V_1(0) \leq 3, -6 \leq V_2(0) \leq 6)$ with parameter values $\phi(0) = 8$, $\alpha = 0.45$, $\beta_R = 0.8$, $\gamma = 5.037$, $k = 3.2$ and $\beta_c = 2.5$.

traveling waves [Fig. 3(d)] via a chimera state [Fig. 3(c)]. The mean phase velocity profile in Fig. 3(b) is completely erratic (complete incoherence), while it is flat in Fig. 3(d) as a proof of complete coherence. Fig. 3(c) shows a peculiar form of coherence-incoherence pattern where we can clearly distinguish between small incoherent parts and coherent parts in which a larger number of nodes are self-organized in the form of traveling waves. The phase chimera nature of this coherence-incoherence state is demonstrated by the mean phase velocity profile in Fig. 3(c), where the nodes in the coherent clusters are characterized by the same mean phase velocity, and the incoherent nodes have disparate phase velocities that break away from the common phase velocity of the coherent nodes. Furthermore, the network nodes belonging to the incoherent parts of the chimera state exemplifies in Fig. 3(c) exhibit irregular oscillations while the nodes of the coherent parts evolve on periodic oscillations [see Fig. 4(a)]. Fig. 4(b) shows that the oscillations of the nodes of the traveling waves pattern illustrated in Fig. 3(d) are periodic.

For an overall view of the dynamical behavior of the considered network of JJSWBOs for the parameter values mentioned above and the control parameter $\sigma$, we resort to the strength of incoherence, a global order parameter that helps to characterize diverse collective states in the coupled system, depending on their degree of incoherence. The strength of incoherence is given as follows Gopal *et al.* (2018):

$$S = 1 - \frac{1}{M} \sum_{m=1}^{M} H(\delta - s_m), \tag{4}$$

where $H(\cdot)$ stands for the Heaviside function, $\delta$ is a small threshold, and

$$s_m = \langle \sqrt{\frac{1}{n} \sum_{i=(m-1)n+1}^{mn} [Z_i - \langle Z \rangle_m]^2} \rangle_t \tag{5}$$

where $n = N/M$, $M$ is the number of bins of nodes of equal size $n$, $m = 1, 2, \ldots, M$,

$$z_i = v_i^1 - v_{i+1}^1, \quad \langle z \rangle_m = \frac{1}{n} \sum_{i=(m-1)n+1}^{mn} z_i, \quad \text{and} \quad \langle \cdot \rangle_t$$

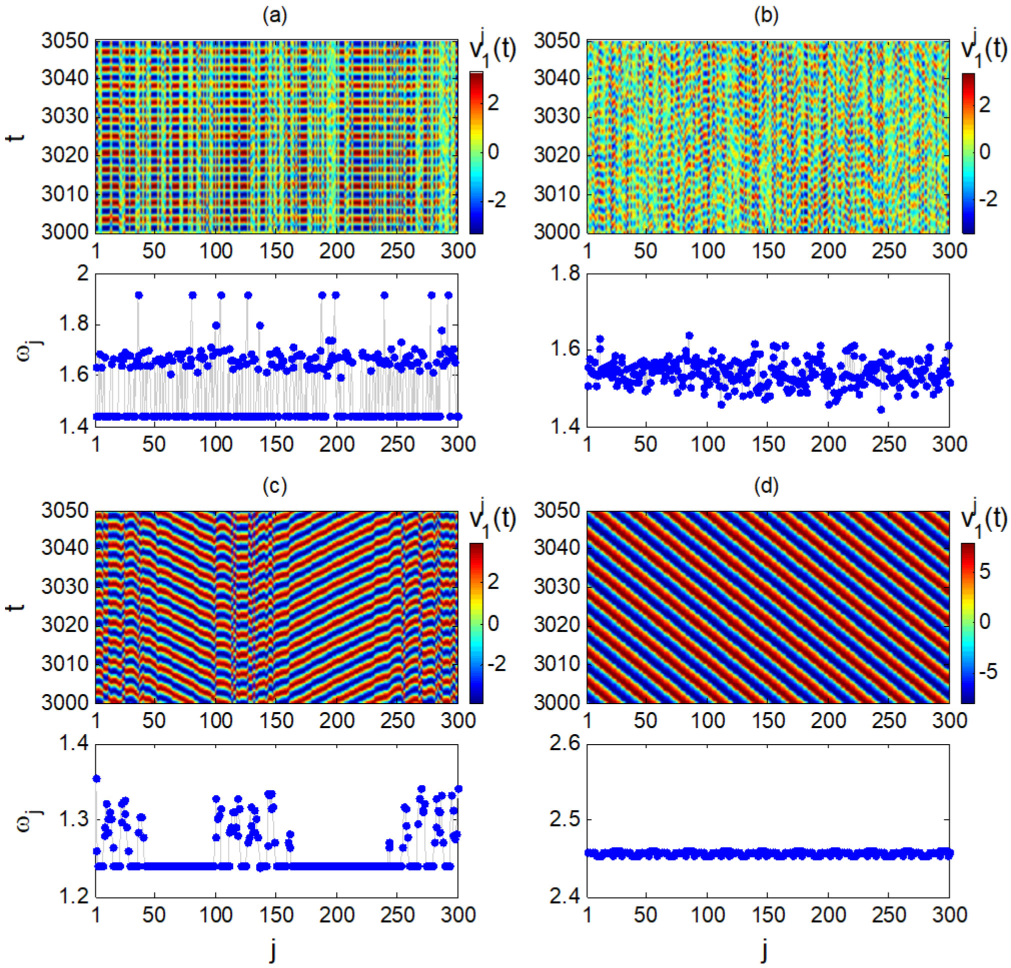


**Figure 3** Prominent collective states exhibited by the network of nonlocally coupled JJSWBOs [Eq. (2)] for $N = 300$, $a = 0.08$, $b = -0.8$, $P = 20$, and different values of $\sigma$ **(a)** $\sigma = 0.00$; **(b)** $\sigma = 0.06$; **(c)** $\sigma = 0.20$; **(d)** $\sigma = 0.60$. Other parameters are given in Fig. 2. In each subfigure, the top panel shows the spatiotemporal plot of the variable $v_1(t)$ for the observed pattern and the bottom panel shows the corresponding phase velocity profile $\{\omega_j, j = 1, 2, \ldots, N\}$.

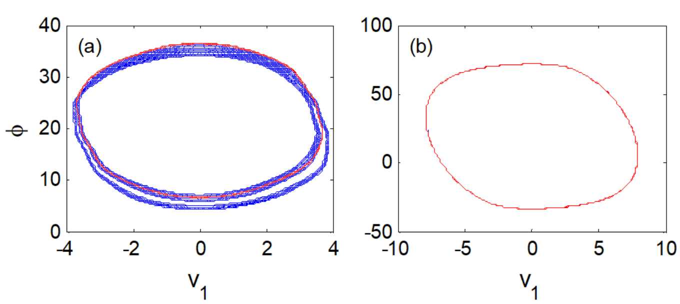


**Figure 4** Phase portraits of two selected nodes ($j = 130$ and $j = 200$) of the chimera pattern and traveling waves pattern shown in Fig. 3: (a) parameters of Fig. 3(c); and (b) parameters of Fig. 3(d). The phase portrait of node 130 (respectively, 200) is displayed in blue (respectively red) line. Note that in (a) node 130 belongs to an incoherent part of the chimera state and node 200 to a coherent part. In (b) the two phase portraits are merged with one another.

stands for the time average. The value of $\delta$ is chosen such that, if the bin $m$ is coherent, then $s_m < \delta$. Overall, $S \to 1$ for completely incoherent states, $S \to 0$ for complete coherent states, and $0 < S < 1$ for coherence-incoherence states, including chimera states. Fig. 5 shows the variation of the strength of incoherence $S$ in the case of the network under study in this paper, when the value of the coupling parameter $\sigma$ is varying. Fig. 5 confirms the transition from complete incoherence ($S \to 1$) to complete coherence ($S = 0$) via chimera states ($0 < S < 1$) when the value of $\sigma$ is increasing.

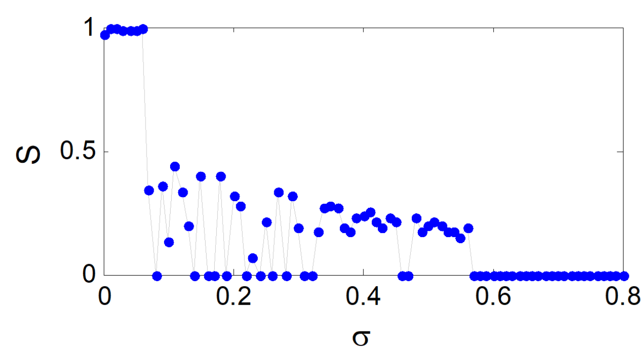


**Figure 5** Strength of incoherence $S$ versus coupling parameter $\sigma$. The parameters are given in Figs. 2 and 3.

## CONCLUSION

This paper dealt with the dynamical exploration of a network of nonlocally coupled Josephson junctions spurred by Wien bridge oscillators. First, the dynamics of an isolated network unit (i.e., a Josephson junction spurred by a Wien bridge oscillator) was investigated numerically, which revealed that the local dynamics is rich, including multistability. For the investigation of the collective dynamics of the considered network, the local dynamics was set to be bistable, involving periodic and chaotic behaviors. The initial conditions were specially prepared such that the network nodes were randomly distributed on these two types of dynamical behaviors. Varying the coupling strength, the nonlocally coupled Josephson junctions spurred by Wien bridge oscillators was found to exhibit complete incoherence, coherent travelling waves and chimera behaviors.

### Availability of data and material

Not applicable.

### Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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