An Investigation into the Interpretation of Pi (π) as the Arithmetic Mean of the Golden Ratio (ϕ) and the Feigenbaum Constant (δ)

İhsan Pehlivan [ID]*,[1], **Selahattin Bulut** [ID]*,α,[2] **and Emin Güney** [ID]β,[3]

*Sakarya University of Applied Sciences, Faculty of Technology, Deparment of Electrical and Electronics Engineering, Sakarya, Türkiye, αSakarya University of Applied Sciences Graduate Education Institute, Sakarya, Türkiye, βSakarya University of Applied Sciences, Faculty of Technology, Department of Computer Engineering, Sakarya, Türkiye.

ABSTRACT This study comprehensively examines the numerical proximity of the arithmetic mean of the golden ratio (ϕ) and the Feigenbaum constant (δ) to the number π, the theoretical mechanisms underlying this relationship, and its epistemological value. Although these three fundamental constants, situated at the intersection of mathematics and theoretical physics, are generally treated as elements of independent disciplines, this research unearths a unique structural bridge expressed through the formulation $\pi = (\phi + \delta)/2$. Numerical analyses prove that the relationship in question can be evaluated within the class of high-precision mathematical approximations, with a notably low relative error margin of approximately %0.064. Within the scope of the study, the fact that the arithmetic balance between the "most irrational number" ϕ, representing maximum dynamic stability, and the universal scaling factor δ, defining the critical threshold at which systems drift into chaos, corresponds to the number π the foundation of periodic cycles is discussed within the framework of the "geometric balance between chaos and order" hypothesis. Whether this numerical pattern is a coincidental numerological coincidence or a yet to be discovered deep topological necessity is analyzed in the context of dynamical systems theory and Euler's identity. In conclusion, by presenting a simple and elegant equation that has gone unnoticed in the half century since the discovery of the Feigenbaum constant, this article raises new epistemological questions regarding the nature of fundamental constants and offers an interdisciplinary perspective on understanding the hidden geometry of nature.

INTRODUCTION

The mathematical and physical constants that define the fundamental workings of the universe point toward a hidden geometry lying deep within nature and the dynamic order beneath chaotic systems. Fundamental constants, situated at the intersection of mathematics and theoretical physics, are considered numerical signatures representing deep structural rules in the fabric of the universe. The number π, the cornerstone of Euclidean geometry; the golden ratio (ϕ), at the heart of growth dynamics and aesthetics; and the Feigenbaum constant (δ), which defines universal scaling in the processes of transition to chaos in non-linear systems despite being discovered in different periods of history converge

unexpectedly within modern dynamical systems theory (Smith 2013; Freinacht 2022).

The Number Pi (π): The Constant of Periodicity and Rotational Symmetry

The number π, defined as the ratio of a circle's circumference to its diameter, is one of the oldest and most well-known mathematical constants in history. With a numerical value of 3.1415926535..., this constant is a transcendental and irrational number, serving as a fundamental determinant of periodic processes not only in circular geometry but also in wave mechanics, quantum physics, and probability theory (Borwein 2000; Agarwal et al. 2013; Eymard and Lafon 2004). Also referred to as "Archimedes' constant" in the literature, π plays a central role in the mathematical expression of harmonic oscillations (for example, the sine and cosine functions). In modern physical theories, π is viewed as a reflection of the isotropic nature of space and rotational symmetry (Singh 2008). Its transcendental and irrational nature causes the decimal digits

of Pi to extend infinitely without ever repeating. From geometry to trigonometry, and from cosmology to statistics, Pi is the fundamental key that expresses the symmetry and periodicity of the universe.

The Golden Ratio (ϕ): Recursive Growth and Dynamic Balance

The golden ratio is an algebraic irrational number expressed by the value:

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.6180339887...$$

Emerging as the limit of the ratio of successive terms in the Fibonacci sequence, this number is observed in spiral structures in nature, plant phyllotaxis, and the geometry of galactic arms. From the perspective of dynamical systems, ϕ is considered the "least rational" or "most irrational" number (Kak 2010; Ho et al. 2015; Akhtaruzzaman and Shafie 2011; Dunlap 1997). This definition stems from the fact that the continued fraction expansion of the golden ratio consists solely of ones:

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + ...}}}$$

By virtue of this property, ϕ represents the state in which a dynamical system is most resistant to chaotic resonances; for it is the value that is most difficult to approximate with rational numbers (Dunlap 1997). The Golden Ratio is an algebraic irrational mathematical constant accepted as an indicator of aesthetic and functional perfection in nature, art, and architecture. From biological growth to crystallography, the golden ratio serves as the coefficient of "optimal balance" across many fields. Defined algebraically as the positive root of the equation $x^2 - x - 1 = 0$, ϕ also emerges as the limit of the ratio of successive numbers in the Fibonacci sequence. The universality of the Golden Ratio has been documented in a vast literature ranging from biology to astrophysics.

The Feigenbaum Constant (δ): The Universal Law of Transition to Chaos

Discovered by Mitchell Feigenbaum in 1975, the mathematical irrational constant $\delta \approx 4.6692016091...$ defines the ratio of intervals in the period doubling bifurcations of non-linear chaotic systems (Smith 2013; Freinacht 2022). In his studies on quadratic maps (for example, the logistic map), Feigenbaum demonstrated that the ratio of the difference between parameter values where a system's stable cycle doubles to the next doubling interval approaches a constant value (δ) (Smith 2013). The truly revolutionary aspect of this discovery is that this ratio remains the same regardless of the system's details (whether the map is a sine function or a polynomial) (Smith 2013). The Feigenbaum constant expresses the "scaling law" in the transition from a regular structure to a chaotic structure in complex processes in the physical world, such as turbulence, cardiac arrhythmias, and population dynamics (Freinacht 2022).

NEW HIDDEN GEOMETRY BETWEEN THE GOLDEN RATIO(ϕ), FEIGENBAUM CONSTANT(δ), AND π

In this section, a hidden geometry discovered between the golden ratio, the Feigenbaum constant, and the number pi, which is not found in the literature, is presented. The fundamental claim is the surprising degree of proximity of the arithmetic mean of the golden ratio (ϕ) and the Feigenbaum constant (δ) to the number (π). The geometry discovered between the golden ratio, the Feigenbaum

constant, and the number pi is formulated as shown in Table 1 and Equation 1;

Table 1 The geometry discovered between the golden ratio(ϕ), the Feigenbaum constant(δ), and the number pi(π)

π	$\dfrac{\phi + \delta}{2}$
3.14159265	3.14**361779**

$$\pi \approx \frac{\phi + \delta}{2} \tag{1}$$

Table 2 Numerical Values and Definitions of the Constants

Variable	Numerical Value	Formula / Definition
Golden Ratio (ϕ)	1.61803398	$\dfrac{1 + \sqrt{5}}{2}$
Feigenbaum Constant (δ)	4.66920160	$\lim\limits_{n \to \infty} \dfrac{\mu_{n+2} - \mu_{n+1}}{\mu_{n+1} - \mu_n}$
Number π	3.14159265	$\dfrac{\text{Circle's Circumference}}{\text{Circle's Diameter}}$
$(\phi + \delta)/2$	3.14361779	Arithmetic Mean
Absolute Difference	0.00202514	Quality of convergence

If a numerical comparison and error margin analysis are performed based on Table 2, it is observed from the results of calculations carried out to 8 decimal places that the absolute difference is at the level of 0.002025. This difference corresponds to a relative error of approximately %0.064. In scientific literature, proximity at this level is generally classified as "high precision". The numerical accuracy and theoretical validity of this correlation should be examined through the mathematical concept of "approximation" (Lange 2010). Is this high-precision numerical proximity merely a "numerological coincidence" with no meaningful connection, or is it a reflection of a yet to be discovered deep mathematical or physical principle? Determining the mathematical nature and the place in science of these three universal constants is critically important for understanding whether the relationship between them is accidental or structural. Each of these constants is defined as an "invariant" within its own discipline and persists regardless of the scale of the system. In the history of physics, the search for numerical relationships such as Bode's Law that are seemingly meaningful but lack a fundamental theoretical basis has a long history. Given the countless existing mathematical and physical constants, the probability of finding such proximities by bringing these constants together in an infinite number of algebraic combinations is statistically expected. This argument suggests that the presented relationship might be nothing more than a random pattern extracted from a vast sea of probabilities.

On the other hand, considering how fundamental and universal each of these three constants is in their respective fields (geometry, growth/ratio, and chaos/universality), the probability that

this simple and elegant relationship between them is a mere co-incidence diminishes significantly. Perhaps these constants are different faces of a more fundamental mathematical structure that unifies them under a single umbrella. This unexpected bridge between geometry, algebra, and dynamical systems may serve as a vital clue regarding the fundamental workings of nature at its most basic level. Some studies in the literature investigating the relationships between transcendental numbers and mathematical constants argue that a deeper topological reason may underlie such relationships, given that the constants under examination are the limit values of recursive processes (Smith 2013). Similarly, this study evaluates the possibility that a profound topological connection exists behind the striking numerical proximity of the arithmetic mean of the golden ratio (ϕ) and the Feigenbaum constant (δ) to the number π.
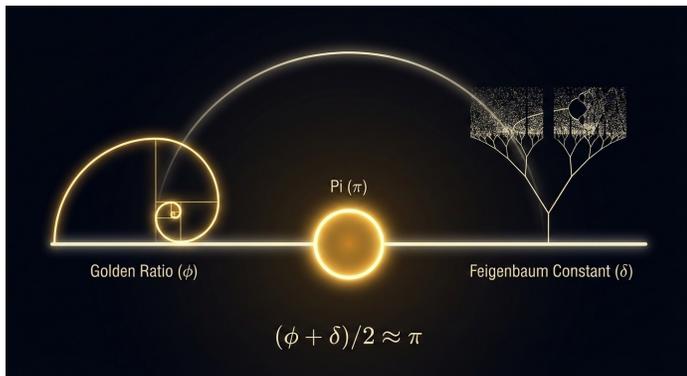

Figure 1 The bridge at the intersection of order (ϕ) and chaos (δ): π.

Figure 1 symbolizes π as the geometric equilibrium between the order of the Golden Ratio (ϕ) and the chaos of the Feigenbaum constant (δ). It illustrates π as a structural bridge unifying these two fundamental dynamics of nature.

It can be argued that these three numbers did not come together by chance, but rather represent a geometry of the transition between chaos and order. The golden ratio is the most irrational number. In dynamical systems (for example, in planetary orbits or KAM theory), it represents the point where a system is most resistant to chaos, that is, where it remains most orderly. The Feigenbaum constant is the universal scaling factor at the point where order completely breaks down and transitions into chaos ("period doubling"). The number π is the foundation of periodic cycles, namely circular motion. The equality Equation 1 may be whispering the following: "The balance point between maximum order (ϕ) and the onset of complete chaos (δ) is the fundamental periodic cycle (π)". By utilizing the equation, the existence of the relationship shown in Table 3 can also be demonstrated.

Table 3 Proximity in the complex plane

$e^{i \cdot 2\pi}$	$e^{i(\phi+\delta)}$
$1.0 + 0.0i$	$0.999991 + 0.00405i$

As can be seen from Table 3, due to the fact that the sum of two very special numbers, such as the golden ratio and the Feigenbaum constant, is nearly equal to the universal value of 2π, it is observed

to satisfy Euler's identity in a surprising manner (with an error on the order of approximately 10^{-6}). This new correlation, which extracts order out of chaos, has revealed that the value 2π the key to circular geometry can be approximately derived from the sum of δ, the universal constant of chaos, and ϕ, the constant of growth.

RESULTS AND DISCUSSION

In this article, an attempt has been made to establish a theoretical bridge concerning how a connection can be formed between π, ϕ, and δ. At the focal point of the study lies the approximate numerical correlation Equation 1, which unifies three fundamental mathematical constants. The high numerical accuracy of this observation has been demonstrated, while the inherent probability of coincidence in such relationships has also been emphasized. It is considered that this numerical relationship, bringing together three constants from seemingly unrelated branches of mathematics, such as geometry (π), universal growth dynamics and aesthetics (ϕ), and the processes of transition to chaos (δ), possesses significant potential for investigation in terms of nature, physics, and mathematics. Such research supports the idea that fundamental constants are not independent but may be interconnected by a deeper mathematical logic. This article has investigated the deep and unexpected connections that may exist between fundamental constants from different fields of mathematics through a critical perspective. Although the Feigenbaum constant has been known for 50 years, the fact that the approximate value of π can be obtained by taking its arithmetic mean with the golden ratio has not been identified by any researcher until now.

CONCLUSION

Some significant contributions of this study include: establishing a bridge between seemingly independent mathematical constants, demonstrating a simple methodology for investigating such apparent patterns, and highlighting the profound epistemological challenges of distinguishing pattern from coincidence. It is evaluated that the hypothesis presented by this simple, elegant, and conspicuous equation may serve as a tool for "intellectual provocation" in terms of raising new research questions regarding the nature of fundamental constants. Systematically examining whether other simple and elegant relationships, like the one proposed in this study, exist between other mathematical and physical constants and discovering potential connections will reduce the probability of the presented hypothesis being a coincidence. While the speculative nature of the ideas presented in the article is acknowledged, it is believed that such interdisciplinary and bold explorations are an absolute necessity for pushing the boundaries of science and understanding the fundamental workings of the universe at its most basic level. It is further evaluated that other correlations to be discovered among universal constants in different disciplines of mathematics and physics may serve as the key to future major scientific breakthroughs.

Ethical standard

The authors have no relevant financial or non-financial interests to disclose.

Availability of data and material

Not applicable.

Conflicts of interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

LITERATURE CITED

Agarwal, R. P., H. Agarwal and S. K. Sen, 2013 Birth, growth and computation of pi to ten trillion digits. Advances in difference equations **2013**: 100.

Akhtaruzzaman, M. and A. A. Shafie, 2011 Geometrical substantiation of Phi, the golden ratio and the baroque of nature, architecture, design and engineering. International Journal of Arts **1**: 1–22.

Borwein, P., 2000 The amazing number Pi. Nieuw Archief Voor Wiskunde **1**: 254–258.

Dunlap, R. A., 1997 The golden ratio and Fibonacci numbers. World Scientific.

Eymard, P. and J. P. Lafon, 2004 The Number π. American Mathematical Soc.

Freinacht, H. (2022). Is Metamodernism the Last Stage of Development? Chaos Theory Might Hold the Answer. *Metamoderna*. https://metamoderna.org/is-metamodernism-the-last-stage-of-development-chaos-theory-might-hold-the-answer/

Ho, M.-W., M. El Naschie and G. Vitiello, 2015 Is spacetime fractal and quantum coherent in the golden mean. Global Journal of Science Frontier Research **15**: 61–80.

Kak, S., 2010 The golden mean and the physics of aesthetics. Ancient Indian leaps into mathematics, Springer: 111–119.

Lange, M., 2010 What are mathematical coincidences (and why does it matter)?. Mind **119**: 307–340.

Singh, R. B., 2008 Introduction to modern physics. New Age International.

Smith, R. D., 2013 Period doubling, information entropy, and estimates for Feigenbaum's constants. International Journal of Bifurcation and Chaos **23**: 1350190.